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Integrable flows in $c = 1$ string theory

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Abstract

In this paper, we review the method of constructing integrable deformations of the compactified $c = 1$ bosonic string theory by primary fields (momentum or winding modes), developed recently in collaboration with S Alexandrov and V Kazakov. The method is based on the formulation of the string theory as a matrix model. The flows generated by either momentum or winding modes (but not both) are integrable and satisfy the Toda lattice hierarchy.

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1. Introduction

The $c = 1$ string theory (the theory of random surfaces embedded in one dimension) can be constructed as the collective theory for a one-dimensional $N \times N$ Hermitian matrix field [1]. The $U(N)$ -invariant sector of the matrix model is described by a non-relativistic quantum mechanics of free fermions in an upside-down quadratic potential. The mapping to free fermions allows us to calculate virtually any quantity in the string theory in all orders in the genus expansion.

The elementary excitations of the $c = 1$ string represent collective excitations of free fermions. The tree-level S -matrix can be extracted by considering the propagation of ‘pulses’ along the Fermi sea and their reflection off the ‘Liouville wall’ [2].

The exact non-perturbative S -matrix has been calculated by Moore *et al* [3]. Each S -matrix element can be associated with a single fermionic loop with a number of external lines. One can then expect that the theory is also solvable in a nontrivial, time-dependent background generated by a finite tachyonic source. Dijkgraaf *et al* [4] demonstrated that this is indeed the case when the allowed momenta form a lattice as in the case of the compactified Euclidean theory. In [4] it has been shown that the string theory compactified at any radius R possesses

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the integrable structure of the Toda lattice hierarchy [5]. The operators associated with the momentum modes in the string theory have been interpreted in [4] as Toda flows.

The explicit construction of these Toda flows is an interesting and potentially important task, because this would allow us to explore the time-dependent string backgrounds. This problem was recently solved in [6]. The method used in [6] is conceptually similar to the method of orthogonal polynomials of Dyson–Mehta in the interpretation of Douglas [7], which has been used to solve the $c < 1$ matrix models [8, 9]. The construction of [6] allows us to evaluate the partition function of the string theory in the presence of finite perturbation by momentum modes. The simplest case of such a perturbation is the sine-Liouville string theory².

Besides the momentum modes, the compactified string theory contains a second type of excitations associated with nontrivial windings around the target circle. In the worldsheet description, the string theory represents a compactified Gaussian field coupled to 2D quantum gravity. Then the momentum and winding modes are the electric and magnetic operators for this Gaussian field. It is known [11, 12] that the winding modes propagate in the non-singlet sector of the $c = 1$ matrix model, which is no more equivalent to a system of fermions. Explicit realization of the winding modes can be given by gauging the compactified matrix model. Then the winding modes are realized as the Polyakov loops winding around the Euclidean time interval [13].

The electric and magnetic operators are exchanged by the usual electric-magnetic duality $R \rightarrow 1/R$ also called T-duality. If the momentum modes of the compactified string theory are described by the Toda integrable flows, then by T-duality the same is true also for the winding modes. A direct proof is presented in [13], where it is also shown that the grand canonical partition function of the matrix model perturbed by only winding modes is a τ -function of the Toda lattice hierarchy. Applying the T-duality backwards, we conclude that the same is true for the partition function of the theory perturbed by only momentum modes.

A special case represents that the theory is compactified at the self-dual radius $R = 1$, it is equivalent to a topological theory that computes the Euler characteristic of the moduli space of Riemann surfaces [14]. When $R = 1$ and only in this case, the partition function of the string theory has alternative realization as a Kontsevich-type model [4, 15].

In this paper, we summarize the results concerning the Toda integrable structure of the compactified $c = 1$ string theory obtained in [6, 13, 16, 17]. We will follow mostly [6], but will present the construction in the framework of the Euclidean compactified theory, while [6] discussed the theory in Minkowski space.

2. The matrix model for the $c = 1$ string theory revisited

2.1. Partition function of the Euclidean theory compactified on a circle with radius R

The $c = 1$ string theory (see the appendix) describes the critical behaviour of the large- N matrix quantum mechanics (see review [1] and the references therein). The critical point is associated with the maximum of the matrix potential. In the scaling limit, which is dominated by dense planar graphs, the potential can be approximated by a quadratic one (with the wrong sign), which is stabilized by imposing a cut-off wall far from the top. The relevant piece of the matrix Hamiltonian is thus

$$H_0 = \frac{1}{2} \text{Tr}(P^2 - M^2) \quad (2.1)$$

² It is conjectured [10] that such a perturbation can lead to a target space geometry with a horizon.

where $P = -i\partial/\partial M$ and M_{ij} is an $N \times N$ Hermitian matrix variable. The cosmological constant μ is introduced as a ‘chemical potential’ coupled to the size of the matrix N , which should be considered as a dynamical variable.

We will consider the Euclidean theory with periodic Euclidean time $x = -it$. The time interval

$$\beta = 2\pi R \tag{2.2}$$

can also be interpreted as inverse temperature. The matrix model describes the $c = 1$ string only when $\beta > \beta_{KT} = 4\pi$. At $\beta = \beta_{KT}$ the winding modes associated with the $N(N - 1)$ angular degrees of freedom become important and produce a Berezinski–Kosterlitz–Thouless-like phase transition to a $c = 0$ string theory [11]. The angular degrees of freedom can be interpreted as the winding modes of the string, which represent strings winding several times around the target circle.

A sensible matrix model for $\beta < \beta_{KT}$ can be constructed by introducing an additional $SU(N)$ gauge field $A(x)$, which projects to the sector free of winding modes [12]. The partition function of the compactified system is

$$\mathcal{Z}(\mu) = \sum_{N \geq 0} e^{-\beta N} \mathcal{Z}_N \tag{2.3}$$

where \mathcal{Z}_N is given by a functional integral with respect to the one-dimensional matrix Hermitian fields $M(x)$ and $A(x)$ with periodic boundary conditions

$$\mathcal{Z}_N = \int_{\text{periodic}} \mathcal{D}P \mathcal{D}M \mathcal{D}A e^{-\frac{1}{\hbar} S_0}. \tag{2.4}$$

The action functional is

$$S_0 = \text{Tr} \int_0^\beta (iP \nabla_A M - H_0) dx \tag{2.5}$$

$\nabla_A M$ denotes the covariant time derivative

$$\nabla_A M = \partial_x M - i[A, M]. \tag{2.6}$$

The gauge field can also be used to study perturbations by winding modes, with appropriately tuned coupling constants, so that the long-range order is not destroyed. It was shown in [13] that the winding modes couple to the moments of the $SU(N)$ magnetic flux associated with the gauge field A .

2.2. Chiral quantization

The analysis of the matrix model simplifies considerably if it is formulated in terms of the chiral variables [6]

$$X_\pm = \frac{M \pm P}{\sqrt{2}} \tag{2.7}$$

representing $N \times N$ Hermitian matrices. The Hamiltonian in the new variables is

$$H_0 = -\frac{1}{2} \text{Tr}(\hat{X}_+ \hat{X}_- + \hat{X}_- \hat{X}_+) \tag{2.8}$$

where the matrix operators \hat{X}_\pm obey the canonical commutation relation

$$[(\hat{X}_+)^j, (\hat{X}_-)^k] = -i\delta_l^j \delta_j^k. \tag{2.9}$$

The partition function for fixed N is now given by a path integral with respect to the one-dimensional Hermitian matrix fields $X_+(x)$, $X_-(x)$ and $A(x)$, satisfying periodic boundary conditions. The action (2.5) reads, in terms of the new fields,

$$S_0 = \text{Tr} \int_0^\beta (iX_+ \nabla_A X_- - \frac{1}{2} X_+ X_-) dx. \tag{2.10}$$

The partition function (2.10) depends on the gauge field only through the global holonomy factor, given by the unitary matrix

$$\Omega = \hat{T} e^{i \int_0^{2\pi R} A(x) dx}. \tag{2.11}$$

This can be seen by imposing the gauge $A = 0$, which can be done at the expense of replacing the periodic boundary condition $X_\pm(\beta) = X_\pm(0)$ by an $SU(N)$ -twisted one

$$X_+(\beta) = X_+(0) \quad X_-(\beta) = \Omega^{-1} X_-(0) \Omega. \tag{2.12}$$

2.3. Gauge-invariant collective excitations: momentum and winding modes

The momentum modes $V_{n/R}$ of the matrix field are gauge-invariant operators with time evolution $V_{n/R}(t) = e^{-nt/R} V_{n/R}(0)$. Here n can be any positive or negative integer. The spectrum of momenta is determined by the condition that the time dependence is periodic in the imaginary direction with period $2\pi R$. The operators satisfying these conditions are

$$V_{n/R} = e^{\pm inx/R} \text{Tr} X_\pm(x)^{n/R} = \begin{cases} \text{Tr} X_+^{|n|/R} & \text{if } n > 0 \\ \text{Tr} X_-^{|n|/R} & \text{if } n < 0. \end{cases} \tag{2.13}$$

This form of the left and the right momentum modes was first suggested to our knowledge by Jevicki [18].

As was shown in [13], the winding modes \tilde{V}_{nR} are associated with the $SU(N)$ magnetic flux through the compactified spacetime

$$\tilde{V}_{nR} = \text{Tr} \Omega^n \tag{2.14}$$

where Ω is the $U(N)$ holonomy (2.11). From the worldsheet point of view the winding operator \tilde{V}_{nR} creates a puncture with a Kosterlitz–Thouless vortex, i.e. a line of discontinuity $2\pi Rn$ starting at the puncture.

2.4. The partition function as a three-matrix integral

In the $A = 0$ gauge the path integral with respect to the fields $X_\pm(x)$ is Gaussian with the determinant of the quadratic form equal to one. Therefore it is reduced to the integral with respect to the initial values $X_\pm = X_\pm(0)$ of the action (2.10) evaluated along the classical trajectories, which satisfy the twisted periodic boundary condition (2.12). Therefore, the canonical partition function of the matrix model can be reformulated [6] as an ordinary matrix integral with respect to the two Hermitian matrices X_+ and X_- , and the unitary matrix Ω :

$$\mathcal{Z}_N = \int DX_+ DX_- d\Omega e^{i \text{Tr}(X_+ X_- - q X_- \Omega X_+ \Omega^{-1})} \tag{2.15}$$

where we denoted

$$q = e^{i\beta} = e^{2\pi i R}. \tag{2.16}$$

A general perturbation by momentum modes can be introduced by changing the homogeneous measures DX_\pm to

$$[DX_\pm] = DX_\pm e^{\pm i \text{Tr} U_\pm(X_\pm)} \tag{2.17}$$

where we introduced the matrix potentials

$$U_{\pm}(X_{\pm}) = R \sum_{n>0} t_{\pm n} \text{Tr} X_{\pm}^{n/R}. \tag{2.18}$$

Similarly, a general perturbation by winding modes can be introduced by changing the invariant measure $D\Omega$ on the group $U(N)$ to

$$[D\Omega] = D\Omega e^{\tilde{U}(q^{1/2}\Omega) - \tilde{U}(q^{-1/2}\Omega)} \tag{2.19}$$

where we introduced the following matrix potential:

$$\tilde{U}(\Omega) = \sum_{n \neq 0} \tilde{t}_n \text{Tr} \Omega^n. \tag{2.20}$$

In the case of a generic perturbation with both momentum and winding modes, the three-matrix model (2.15) is not integrable. However, it can be solved exactly in the case of an arbitrary perturbation only by momentum or only by winding modes. In the following, we will discuss in detail these two integrable cases. In both cases it is possible to integrate with respect to the angles and the matrix partition function reduces to an eigenvalue integral.

2.5. Integration with respect to the angles

Let us consider the case of a perturbation with only momentum modes. Then the gauge field plays the role of a Lagrange multiplier for the condition $[X_+, X_-] = 0$ and the two matrices can be simultaneously diagonalized. Applying twice the Harish–Chandra–Itzykson–Zuber formula we can reduce the matrix integral to an integral over the eigenvalues $x_1^{\pm}, \dots, x_N^{\pm}$ of the two Hermitian matrices X_{\pm} . We write, after rescaling the integration variables,

$$\mathcal{Z}_N(t) = \int_{-\infty}^{\infty} \prod_{k=1}^N [dx_k^+] [dx_k^-] \det_{jk} (e^{ix_j^- x_k^+}) \det_{jk} (e^{-iqx_j^- x_k^+}) \tag{2.21}$$

where the measures are defined by

$$[dx_{\pm}] = dx_{\pm} e^{iU_{\pm}(x_{\pm})}. \tag{2.22}$$

Then the grand canonical partition function can be written as a Fredholm determinant

$$Z(\mu, t) = \text{Det}(1 + e^{-\beta\mu} K_+ K_-) \tag{2.23}$$

where

$$[K_+ f](x_-) = \int [dx_+] e^{ix_+ x_-} f(x_+) \quad [K_- f](x_+) = \int [dx_-] e^{-iqx_+ x_-} f(x_-). \tag{2.24}$$

3. The partition function of the non-perturbed theory

3.1. Eigenfunctions of the non-perturbed Hamiltonian

In the absence of perturbation, the partition function is given by the Fredholm determinant

$$\mathcal{Z}(\mu) = \text{Det}(1 + e^{-\beta(\mu + H_0)}) \tag{3.1}$$

and can be interpreted as the grand canonical finite-temperature partition function for a system of non-interacting fermions in the inverse Gaussian potential. The Fredholm determinant (3.1) can be computed once we know a complete set of eigenfunctions for the one-particle Hamiltonian

$$H_0 = -\frac{1}{2}(\hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+). \tag{3.2}$$

The canonically conjugated operators \hat{x}_+ and \hat{x}_-

$$\hat{x}_+\hat{x}_- - \hat{x}_-\hat{x}_+ = -i \quad (3.3)$$

are represented in the space of functions of x_+ as

$$\hat{x}_+ = x_+ \quad \hat{x}_- = i\partial_{x_+}$$

and in the space of functions of x_- as

$$\hat{x}_- = x_- \quad \hat{x}_+ = -i\partial_{x_-}.$$

Since the inverse Gaussian potential is bottomless, the spectrum of the Hamiltonian is continuous

$$H_0\psi_+^E(x_\pm) \equiv \mp i(x_\pm\partial_{x_\pm} + 1/2)\psi_+^E(x_\pm) = E\psi_+^E(x_\pm) \quad (E \in \mathbb{R}). \quad (3.4)$$

The eigenfunctions in the x_\pm -representation, which we denote by Dirac brackets

$$\psi_\pm^E(x_\pm) = \langle E|x_\pm \rangle$$

are given by

$$\langle E|x_\pm \rangle = \frac{1}{\sqrt{\pi}} e^{\mp \frac{1}{2}i\phi_0 x_\pm^{\pm iE - \frac{1}{2}}} \quad (3.5)$$

where the phase factor $\phi_0 = \phi_0(E)$ will be determined in a moment³. The solutions (3.5) have a branch point at $x_\pm = 0$ and are defined unambiguously only on the positive real axis. For each energy E there are two solutions, since there are two ways to define the analytic continuation to the negative axis. The wavefunctions relevant to our problem have large negative energies and are supported, up to exponentially small terms due to tunnelling phenomena, by the positive axis⁴. Therefore we define the analytic continuation to negative axis as

$$\begin{aligned} \langle E|-x_+ \rangle &= \langle E|e^{-i\pi}x_+ \rangle = e^{\pi E}\langle E|x_+ \rangle & (x_+ > 0) \\ \langle E|-x_- \rangle &= \langle E|e^{i\pi}x_- \rangle = e^{\pi E}\langle E|x_- \rangle & (x_- > 0). \end{aligned} \quad (3.6)$$

The functions (3.5) with E real form two orthonormal sets

$$\int_{-\infty}^{\infty} dx_\pm \langle E|x_\pm \rangle \langle x_\pm|E' \rangle = \delta(E - E'). \quad (3.7)$$

We will also impose the bi-orthogonality condition

$$\int \int_{-\infty}^{\infty} \frac{dx_+ dx_-}{\sqrt{2\pi}} \langle E|x_+ \rangle e^{ix_+ x_-} \langle x_-|E' \rangle = \delta(E - E') \quad (3.8)$$

which fixes the phase ϕ_0

$$e^{i\phi_0(E)} = \sqrt{\frac{1}{2\pi}} e^{-\frac{\pi}{2}(E-i/2)} \Gamma(iE + 1/2). \quad (3.9)$$

The phase defined by (3.9) has in fact a small imaginary part, which we neglect⁵. This is because the theory restricted to the Hilbert space spanned on these states is strictly speaking not unitary due to the tunnelling phenomena across the top of the potential. In the following, we will systematically neglect the exponentially small terms $\mathcal{O}(e^{\pi E})$. With this accuracy we can write the completeness condition

$$\int_{-\infty}^{\infty} dE \langle x_+|E \rangle \langle E|x_- \rangle = \frac{1}{\sqrt{2\pi}} e^{-ix_+ x_-} \quad (3.10)$$

³ The eigenfunctions (3.5) play the same role in the chiral quantization as the parabolic cylinder functions in the standard quantization, based on the original Hamiltonian (2.1).

⁴ This corresponds to the theory of type I of [3].

⁵ The pure phase factor would be $\sqrt{\frac{\cosh \pi E}{2\pi}} \Gamma(iE + 1/2) = \sqrt{1 + e^{\pi E}} e^{i\phi_0(E)}$.

where the right-hand side is the kernel of the inverse Fourier transformation relating the x_+ and x_- representations. For real energy the phase ϕ_0 is real (up to non-perturbative terms), but we will analytically continue it to the whole complex plane. In this case the reality condition means

$$\overline{\phi_0(E)} = \phi_0(\bar{E}). \tag{3.11}$$

3.2. Cut-off prescription, density of states and free energy

To find the density of states, we need to introduce a cut-off Λ such that $\Lambda \gg \mu$. This can be done by putting a completely reflecting wall at distance $\frac{x_+ + x_-}{\sqrt{2}} = \sqrt{2\Lambda}$ from the origin. Since the wall is completely reflecting, there is no flow of momentum through it, $\frac{x_+ - x_-}{\sqrt{2}} = 0$. Thus the cut-off wall is equivalent to the following boundary condition at $x_+ = x_- = \sqrt{\Lambda}$:

$$\psi_+^E(\sqrt{\Lambda}) = \psi_-^E(\sqrt{\Lambda}) \tag{3.12}$$

on the wavefunctions (3.5). This condition is satisfied for a discrete set of energies $E_n (n \in \mathbb{Z})$ defined by

$$\phi_0(E_n) - E_n \log \Lambda + 2\pi n = 0. \tag{3.13}$$

From (3.13) we can find the density of the energy levels in the confined system:

$$\rho(E) = \frac{\log \Lambda}{2\pi} - \frac{1}{2\pi} \frac{d\phi_0(E)}{dE}. \tag{3.14}$$

Now we can calculate free energy $\mathcal{F}(\mu, R) = \log \mathcal{Z}(\mu, R)$ as

$$\mathcal{F}(\mu, R) = \int_{-\infty}^{\infty} dE \rho(E) \log[1 + e^{-\beta(\mu+E)}] \tag{3.15}$$

with the density (3.14). Integrating by parts in (3.15) and dropping out the Λ -dependent part, we get

$$\mathcal{F}(\mu, R) = -\frac{1}{2\pi} \int d\phi_0(E) \log(1 + e^{-\beta(\mu+E)}) = -R \int_{-\infty}^{\infty} dE \frac{\phi_0(E)}{1 + e^{\beta(\mu+E)}}. \tag{3.16}$$

We close the contour of integration in the upper half plane and take the integral as a sum of residues. This gives for the free energy

$$\mathcal{F} = -i \sum_{r=n+\frac{1}{2} > 0} \phi_0(ir/R - \mu). \tag{3.17}$$

Using the explicit form (3.9) of ϕ_0 we can represent the free energy as a sum over pairs of positive half-integers

$$\mathcal{F}(\mu, R) = \sum_{r,s \geq 1/2}^{\infty} \log(\mu + ir + is/R) \tag{3.18}$$

which is explicitly invariant under the T-duality $R \rightarrow 1/R, \mu \rightarrow R\mu$. From here it follows that the free energy satisfies the functional equation

$$4 \sin(\partial_\mu/2R) \sin(\partial_\mu/2) \mathcal{F}(\mu, R) = -\log \mu. \tag{3.19}$$

From (3.17) we can express the phase factor in terms of the free energy

$$\phi_0 = -2 \sin(\partial_\mu/2R) \mathcal{F}(\mu, R). \tag{3.20}$$

3.3. Calculation of the free energy using the E -representation of the operators \hat{x}_\pm

Let us now give an alternative, algebraic derivation of the free energy of the non-perturbed theory. For this purpose we will write the canonical commutation relation (3.3) for the matrix elements of the operators \hat{x}_\pm in E -representation.

The action of the operators \hat{x}_\pm on the wavefunctions is equivalent to a shift of the energy by an imaginary unit (we consider the wavefunctions as analytic functions of E) and a multiplication by a phase factor:

$$\langle E | \hat{x}_\pm | x_\pm \rangle = e^{\pm \frac{1}{2} i [\phi_0(E \mp i) - \phi_0(E)]} \langle E \mp i | x_\pm \rangle. \tag{3.21}$$

As a consequence, the operators \hat{x}_\pm are represented by the finite-difference operators acting in the E -space

$$\hat{x}_\pm \rightarrow e^{\mp \frac{1}{2} i \phi} e^{\mp i \partial_E} e^{\pm \frac{1}{2} i \phi}. \tag{3.22}$$

The Heisenberg relation (3.3) is equivalent to the condition

$$\phi_0\left(E + \frac{1}{2}i\right) - \phi_0\left(E - \frac{1}{2}i\right) = -\log(-E) \tag{3.23}$$

which is equivalent to the functional constraint (3.19).

4. Integrable perturbations by momentum modes

4.1. One-particle eigenfunctions and the density of states of deformed system

Now we will consider perturbations generated by momentum modes $V_{n/R}$. As we argued in section 2.4, this can be achieved by deforming the integration measures dx_\pm to

$$[dx_\pm] = dx_\pm \exp(\pm i U_\pm(x_\pm)) \quad U_\pm(x_\pm) = R \sum_{k \geq 1} t_{\pm k} x_\pm^{k/R}. \tag{4.1}$$

In this section, we will show that such a deformation is exactly solvable, being generated by a system of commuting flows H_n associated with the coupling constants $t_{\pm n}$. The associated integrable structure is that of a constrained Toda lattice hierarchy. The method is very similar to the standard Lax formalism of Toda theory, but we will not assume that the reader is familiar with this subject. It is based on the possibility of describing the perturbation by vertex operators as deformations of the E -representation of the canonical commutation relation.

The partition function of the perturbed system is given by the Fredholm determinant (2.23), where the integration kernels (2.23) and (2.24) are defined with the deformed measures (4.1). The deformed kernel is diagonalized by a complete orthonormal system of wavefunctions⁶ $\Psi_\pm^E(x_\pm)$

$$\int dx_\pm \overline{\Psi_\pm^E(x_\pm)} \Psi_\pm^E(x_\pm) = \delta(E - E') \tag{4.2}$$

which satisfy the following three conditions:

(1) they are eigenfunctions of the evolution operator relating the points $x = 0$ and $x = \beta$

$$e^{-\beta H_0} \Psi_\pm^E = e^{-\beta E} \Psi_\pm^E \tag{4.3}$$

(2) they behave at infinity as

$$\Psi_\pm^E(x_\pm) \sim x_\pm^{\pm i E - \frac{1}{2}} e^{\mp \frac{1}{2} i \phi(E)} e^{i U_\pm(x_\pm)} \tag{4.4}$$

⁶ As before, we understand the completeness in a weak sense, i.e. up to non-perturbative terms due to tunnelling phenomena, which we have neglected.

(3) they satisfy the bi-orthogonality condition (3.8)

$$\iint \frac{dx_+ dx_-}{\sqrt{2\pi}} \overline{\Psi_-^E(x_-)} e^{ix_+ x_-} \Psi_+^{E'}(x_+) = \delta(E - E'). \tag{4.5}$$

The new eigenfunctions are not necessarily eigenvalues of the Hamiltonian H_0 itself. Indeed, the evolution operator $e^{-\beta H_0}$ acts trivially on any entire function of $x_{\pm}^{1/R}$. In particular, condition (4.4) is compatible with condition (4.3).

Once the basis of the perturbed wavefunctions is found, the logarithm of the Fredholm determinant can be calculated as the integral (3.15), where the new density of states is obtained from the phase $\phi(E)$ in the asymptotics (4.4)

$$\rho(E) = -\frac{1}{2\pi} \frac{d\phi}{dE}. \tag{4.6}$$

Therefore the phase $\phi(E)$ contains all the information about the perturbed system. It is related to the free energy by

$$\mathcal{F} = -i \sum_{r \geq 1/2} \phi(ir/R - \mu). \tag{4.7}$$

$$\phi(-\mu) = 2 \sin(\hbar \partial_{\mu}/2R) \mathcal{F}(\mu, R). \tag{4.8}$$

4.2. Dressing operators

Now we proceed to the actual calculation of the phase ϕ . The method is a generalization of the algebraic method we have used in section 3.3. First we remark that the perturbed wavefunctions are given by the rhs of (4.4) up to factors of the form

$$W_{\pm}(x_{\pm}) = \exp \left(iR \sum_{n \geq 1} v_{\pm n} x_{\pm}^{-n/R} \right). \tag{4.9}$$

which satisfy the second condition (4.3) and tend to 1 when $x_{\pm} \rightarrow \infty$. The unknown coefficients v_n and the phase ϕ are determined functions of E and the couplings t_n by the third condition (4.5).

It follows from (3.22) that the deformed wavefunction

$$\Psi_{\pm}^E(x_{\pm}) = e^{iU_{\pm}(x_{\pm})} e^{\mp \frac{1}{2} i\phi(E)} x_{\pm}^{\pm iE - \frac{1}{2}} W_{\pm}(x_{\pm}) \tag{4.10}$$

can be obtained from the bare wavefunctions (3.5) by the action of two finite-difference operators in the E -space $\hat{\mathcal{W}}_+$ and $\hat{\mathcal{W}}_-$:

$$\Psi_{\pm}^E(x_{\pm}) = \langle E | e^{\pm \frac{1}{2} i\phi_0} \hat{\mathcal{W}}_{\pm} | x_{\pm} \rangle. \tag{4.11}$$

The explicit form of the dressing operators $\hat{\mathcal{W}}_{\pm}$ is obtained by replacing $x_{\pm} \rightarrow e^{\mp i\partial_E}$ in (4.10).

The dressing operators are unitary

$$\hat{\mathcal{W}}_+ \hat{\mathcal{W}}_+^{\dagger} = \mathcal{W}_- \hat{\mathcal{W}}_-^{\dagger} = 1 \tag{4.12}$$

since they relate two orthonormal systems of functions. Further, the bi-orthogonality condition (4.5) is equivalent to the identity

$$\hat{\mathcal{W}}_-^{\dagger} e^{i\phi_0} \hat{\mathcal{W}}_+ = 1. \tag{4.13}$$

The identity (4.13) means that the product $\hat{\mathcal{W}}_- \hat{\mathcal{W}}_+^{-1}$ does not depend on the perturbation, which is a general property of all Toda lattice systems [5].

4.3. Lax operators and string equation

We have seen that the partition function is expressed in terms of the phase $\phi(-\mu)$. Therefore we assume that the energy is at the Fermi level $E = -\mu$, and consider all variables as functions of μ instead of E . Let us denote by $\hat{\omega}$ the operator

$$\hat{\omega} = e^{i\partial_\mu} \tag{4.14}$$

shifting the variable μ by i . The operators $\hat{\omega}$ and μ satisfy the Heisenberg–Weyl commutation relation

$$[\hat{\omega}, \mu] = \hat{\omega} \quad [\hat{\omega}^{-1}, \mu] = -\hat{\omega}^{-1}. \tag{4.15}$$

Now let us consider the representation of these commutation relations in the perturbed theory. The dressing operators \mathcal{W}_\pm are now exponents of a series in $\hat{\omega}$ with μ -dependent coefficients

$$\hat{\mathcal{W}}_\pm = e^{iR \sum_{n \geq 1} t_{\pm n} \hat{\omega}^{n/R}} e^{\mp \frac{1}{2} i \phi(\mu)} e^{iR \sum_{n \geq 1} v_{\pm n}(\mu) \hat{\omega}^{-n/R}}. \tag{4.16}$$

The operators

$$L_+ = \mathcal{W}_+ \hat{\omega} \mathcal{W}_+^{-1} \quad L_- = \mathcal{W}_- \hat{\omega}^{-1} \mathcal{W}_-^{-1} \quad M_+ = \mathcal{W}_+ \mu \mathcal{W}_+^{-1} \quad M_- = \mathcal{W}_- \mu \mathcal{W}_-^{-1}. \tag{4.17}$$

known as Lax and Orlov–Schulman operators satisfy the same commutation relations as the operators $\hat{\omega}$ and μ

$$[L_+, M_+] = iL_+ \quad [L_-, M_-] = -iL_-. \tag{4.18}$$

The Lax operators L_\pm represent the canonical coordinates \hat{x}_\pm in the basis of perturbed wavefunctions,

$$\langle E | e^{\pm \frac{1}{2} i \phi_0} \hat{\mathcal{W}}_\pm L_\pm | x_\pm \rangle = x_\pm \langle E | e^{\pm \frac{1}{2} i \phi_0} \hat{\mathcal{W}}_\pm \hat{x}_\pm | x_\pm \rangle \tag{4.19}$$

while the Orlov–Shulman operators M_\pm represent Hamiltonian $H_0 = -\frac{1}{2}(\hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+)$. Therefore, the L and M operators are related also by

$$M_+ = M_- = \frac{1}{2}(L_+ L_- + L_- L_+). \tag{4.20}$$

The last identity is not satisfied automatically in the Toda system and represents an additional constraint analogous to the string equations in the minimal models of 2D quantum gravity. Relation (4.20) is proved by inserting the identity (4.13) in the bare relation for $E = -\mu$, satisfied by the operators (3.22). The string equation can also be written as the Heisenberg commutation relation between the two Lax operators

$$[L_+, L_-] = -i \tag{4.21}$$

which also follows directly from (3.3).

The operators M_\pm can be expanded as an infinite series of the L -operators. Indeed, as they act on the dressed wavefunctions as

$$\begin{aligned} \langle E | e^{\pm \frac{1}{2} i \phi_0} \hat{\mathcal{W}}_\pm M_\pm | x_\pm \rangle &= \pm i(x_\pm \partial_{x_\pm} - 1/2) \Psi_\pm^E(x_\pm) \\ &= \left(\sum_{k \geq 1} k t_{\pm k} x_\pm^{k/R} + \mu + \sum_{k \geq 1} v_{\pm k} x_\pm^{-k/R} \right) \Psi_\pm^E(x_\pm). \end{aligned} \tag{4.22}$$

We can write

$$M_\pm = \sum_{k \geq 1} k t_{\pm k} L_\pm^{k/R} + \mu + \sum_{k \geq 1} v_{\pm k} L_\pm^{-k/R}. \tag{4.23}$$

In order to exploit the Lax equations (4.18) and the string equations (4.20) we need the explicit form of the two Lax operators. It follows from (4.16) that L_{\pm} can be represented as a series of the form

$$\begin{aligned}
 L_+ &= e^{-i\phi/2} \left(\hat{\omega} + \sum_{k \geq 1} a_k \hat{\omega}^{1-n/R} \right) e^{i\phi/2} \\
 L_- &= e^{i\phi/2} \left(\hat{\omega}^{-1} + \sum_{k \geq 1} a_{-k} \hat{\omega}^{-1+n/R} \right) e^{-i\phi/2}.
 \end{aligned}
 \tag{4.24}$$

4.4. Integrable flows

Let us identify the integrable flows associated with the coupling constants t_n . From definition (4.17) we have

$$\partial_{t_n} L_{\pm} = [H_n, L_{\pm}]
 \tag{4.25}$$

where the operators H_n are related to the dressing operators as

$$H_n = (\partial_{t_n} \mathcal{W}_+) \mathcal{W}_+^{-1} = (\partial_{t_n} \mathcal{W}_-) \mathcal{W}_-^{-1}.
 \tag{4.26}$$

The two representations of the flows H_n are equivalent by virtue of relations (4.12) and (4.13). A more explicit expression in terms of the Lax operators is derived by the following standard argument. Let us consider the case $n > 0$. From the explicit form of the dressing operators it is clear that $H_n = W_+ \hat{\omega}^{n/R} W_+^{-1} +$ negative powers of $\hat{\omega}^{1/R}$. The variation of t_n will change only the coefficients of the expansions (4.24) of the Lax operators, preserving their general form. But it is clear that if the expansion of H_n contained negative powers of $\hat{\omega}^{1/R}$, its commutator with L_- would create extra powers $\hat{\omega}^{-1-k/R}$. Therefore

$$H_{\pm n} = (L_{\pm}^{n/R})_{\geq} + \frac{1}{2} (L_{\pm}^{n/R})_0 \quad n > 0
 \tag{4.27}$$

where the symbol $(\)_{\geq}$ means the positive (negative) parts of the series in the shift operator $\hat{\omega}^{1/R}$ and $(\)_0$ means the constant part. By a similar argument one shows that the Lax equations (4.25) are equivalent to the zero-curvature conditions

$$\partial_{t_m} H_n - \partial_{t_n} H_m - [H_m, H_n] = 0.
 \tag{4.28}$$

Equations (4.25) and (4.27) imply that the perturbed theory possesses the Toda lattice integrable structure. The Toda structure implies an infinite hierarchy of PDEs for the coefficients v_n of the dressing operators, the first of which is the Toda equation for the phase $\phi(\mu) \equiv \phi(E = -\mu)$

$$i \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_{-1}} \phi(\mu) = e^{i\phi(\mu) - i\phi(\mu - i/R)} - e^{i\phi(\mu + i/R) - i\phi(\mu)}.
 \tag{4.29}$$

The uniqueness of the solution is assured by appropriate boundary conditions [13], which are equivalent to the constraint (4.21).

4.5. Representation in terms of a bosonic field

The momentum modes can be described as the oscillator modes of a bosonic field $\varphi(x_+, x_-) = \varphi_+(x_+) + \varphi_-(x_-)$. The bosonization formula is

$$\Psi_{\pm}^{E = -\mu - i/2}(x_{\pm}) = \mathcal{Z}^{-1} e^{\pm i\varphi_{\pm}(x_{\pm})} \cdot \mathcal{Z}
 \tag{4.30}$$

where \mathcal{Z} is the partition function and

$$\varphi_{\pm}(x_{\pm}) = R \sum_{k \geq 1} t_k x_{\pm}^{k/R} + \frac{1}{R} \partial_{\mu} + \mu \log x_{\pm} - R \sum_{k \geq 1} \frac{1}{k} x_{\pm}^{-k/R} \frac{\partial}{\partial t_k}. \quad (4.31)$$

Then by (4.22) the operators M_{\pm} are represented by the currents $x_{\pm} \partial_{\pm} \varphi$:

$$M_{\pm}^{\dagger} \Psi_{\pm}^E(x_{\pm})|_{E=-\mu-i/2} = \mathcal{Z}^{-1} x_{\pm} \partial_{\pm} \varphi \cdot \mathcal{Z}. \quad (4.32)$$

4.6. The dispersionless (quasiclassical) limit

Let us reintroduce the Planck constant by replacing $\mu \rightarrow \mu/\hbar$ and consider the quasiclassical limit $\hbar \rightarrow 0$. In this limit the integrable structure described above reduces to the dispersionless Toda hierarchy [5, 19, 20], where the operators μ and $\hat{\omega}$ can be considered as a pair of classical canonical variables with Poisson bracket:

$$\{\omega, \mu\} = \omega. \quad (4.33)$$

Similarly, all operators become c -functions of these variables. The Lax operators can be identified with the classical phase space coordinates x_{\pm} , which satisfy

$$\{x_+, x_-\} = 1. \quad (4.34)$$

The two functions $x_{\pm}(\omega, E)$ define the classical phase space trajectories as functions of the proper time variable $\tau = \log \omega$.

The shape of the Fermi sea is determined by the classical trajectory corresponding to the Fermi level $E = -\mu$. In the non-perturbed system the classical trajectory is

$$x_+(\omega) = \sqrt{\mu} \omega \quad x_-(\omega) = \sqrt{\mu} \omega^{-1} \quad (4.35)$$

and the Fermi sea has a hyperbolic shape

$$x_+ x_- = \mu. \quad (4.36)$$

In the perturbed theory the classical trajectories are of the form

$$x_{\pm} = L_{\pm}(\omega, \mu) \quad (4.37)$$

where the functions L_{\pm} are of the form

$$L_{\pm}(\omega, \mu) = e^{\frac{1}{2} \partial_{\mu} \phi} \omega^{\pm 1} \left(1 + \sum_{k \geq 1} a_{\pm k}(\mu) \omega^{\mp k/R} \right). \quad (4.38)$$

The flows H_n become Hamiltonians for the evolution along the ‘times’ t_n . The unitary operators \mathcal{W}_{\pm} become a pair of canonical transformations between the variables ω, μ and L_{\pm}, M_{\pm} . Their generating functions are given by the expectation values $S_{\pm} = \mathcal{Z}^{-1} \varphi_{\pm}(x_{\pm}) \cdot \mathcal{Z}$ of the chiral components of the bosonic field ϕ :

$$S_{\pm} = \pm R \sum_{k \geq 1} t_{\pm k} x_{\pm}^{k/R} + \mu \log x_{\pm} - \frac{1}{2} \phi \pm R \sum_{k \geq 1} \frac{1}{k} v_k x_{\pm}^{-k/R} \quad (4.39)$$

where $v_k = \partial \mathcal{F} / \partial t_k$. The differential of the function S_{\pm} is

$$dS_{\pm} = M_{\pm} d \log x_{\pm} + \log \omega d\mu + R \sum_{n \neq 0} H_n dt_n. \quad (4.40)$$

If we consider the coordinate ω as a function of either x_+ or x_- , then

$$\omega = e^{\partial_{\mu} S_+(x_+)} = e^{\partial_{\mu} S_-(x_-)}. \quad (4.41)$$

The classical string equation

$$x_+x_- = M_+ = M_- \tag{4.42}$$

can be written, using the expansion (4.23) of M_{\pm} , as

$$\begin{aligned} x_+x_- &= \sum_{k \geq 1} kt_k x_+^{k/R} + \mu + \sum_{k \geq 1} v_k x_+^{-k/R} \\ x_+x_- &= \sum_{k \geq 1} kt_{-k} x_-^{k/R} + \mu + \sum_{k \geq 1} v_{-k} x_-^{-k/R}. \end{aligned} \tag{4.43}$$

The first of these expansions is convergent for sufficiently large x_+ and the second one for sufficiently large x_- . Comparing the two equations one can extract the form of the Fermi surface. Technically this is done as follows [17]. First, note that if all $t_{\pm k}$ with $k > n$ vanish, the sum in (4.38) can be restricted to $k \leq n$. Then it is enough to substitute the expressions (4.38) in the profile equations (4.43) and compare the coefficients in front of $\omega^{\pm k/R}$.

The two expansions (4.43) can be combined into a single equation

$$x_-x_- - \sum_{k \neq 0} kt_k H_k(\omega) = \mu \tag{4.44}$$

where

$$H_{\pm k}(\omega) = [L_{\pm}^{k/R}(\omega)]_{\pm} + \frac{1}{2}[L_{+}^{k/R}(\omega)]_0 \quad (k > 0). \tag{4.45}$$

The left-hand side can be interpreted as the Hamiltonian for the perturbed system. It defines the profile of the perturbed Fermi sea, which is a deformation of the hyperbole (4.36).

4.7. One-point correlators in the dispersionless limit

It follows from $\log \omega = \partial_{\mu} S_{\pm}(x_{\pm})$ that the μ -derivative of the one-point correlators

$$\langle \text{Tr } X_{\pm}^{n/R} \rangle = \frac{\partial \mathcal{F}}{\partial t_n} \tag{4.46}$$

is given by the contour integrals

$$\frac{\partial^2 \mathcal{F}}{\partial \mu \partial t_{\pm n}} = \frac{1}{2\pi i} \oint d\omega^{1/R} [L_{\pm}(\omega)]^{n/R} \quad (n \geq 1) \tag{4.47}$$

where the closed contour of integration in the variable $\omega^{1/R}$ goes along the arc between $\omega = e^{-i\pi R}$ and $\omega = e^{i\pi R}$. (Note that the integrand is expanded in Laurent series in $\omega^{1/R}$.)

4.8. Example: sine-Gordon field coupled to 2D gravity

The simplest nontrivial string theory with time-dependent background is the sine-Gordon theory coupled to gravity known also as sine-Liouville theory. It is obtained by perturbing with the lowest couplings t_1 and t_{-1} . In this case

$$x_{\pm} = e^{\frac{1}{2}\partial_{\mu}\phi} \omega^{\pm 1} (1 + a_{\pm 1} \omega^{\mp \frac{1}{R}}) \tag{4.48}$$

and (4.43) gives

$$\mu e^{-\partial_{\mu}\phi} - \left(1 - \frac{1}{R}\right) t_1 t_{-1} e^{-(2-\frac{1}{R})\partial_{\mu}\phi} = 1 \quad a_{\pm 1} = t_{\mp 1} e^{-\frac{1}{2}(2-\frac{1}{R})\partial_{\mu}\phi}. \tag{4.49}$$

The first equation is an algebraic equation for the susceptibility

$$u_0 = \partial_{\mu}^2 \mathcal{F} = -R \partial_{\mu} \phi.$$

It was first found (for the T-dual theory) in [13]. This algebraic equation resumes the perturbative expansion found in [21]. Equation (4.48) with a_{\pm} given in (4.49) was first found in [16] by integrating the Hirota equations for the Toda hierarchy with the boundary condition given by the non-perturbed free energy (3.18).

5. Integrable perturbations by winding modes

5.1. The partition function as integral over the $U(N)$ group

Let us consider a perturbation of the $c = 1$ string theory by only winding modes. The relevant variable in this case is the gauge field $A(x)$ and the matrix model (2.4), which can be viewed as a two-dimensional gauge theory defined on the disc having as a boundary the compactified Euclidean time interval. The only nontrivial degree of freedom in such a theory is the holonomy factor around the circle (2.11). The canonical partition function is given by the $U(N)$ integral

$$\mathcal{Z}_N = \int \frac{[D\Omega]}{|q^{-1/2}\Omega \otimes I - q^{1/2}I \times \Omega|} \tag{5.1}$$

where I is the unit $N \times N$ matrix. The integrand depends only on the eigenvalues z_1, \dots, z_N of the unitary matrix Ω and the partition function becomes

$$\mathcal{Z}_N(\tilde{t}_n) = |q^{1/2} - q^{-1/2}|^{-N} \oint \prod_{k=1}^N \oint \frac{[dz_k]}{2\pi i z_k} \prod_{j < k} \left| \frac{z_j - z_k}{q^{-1/2}z_j - q^{1/2}z_k} \right|^2. \tag{5.2}$$

where the integration goes along the unit circle $|z| = 1$ with measure

$$[dz] = dz e^{\tilde{U}(q^{1/2}z) - \tilde{U}(q^{-1/2}z)}. \tag{5.3}$$

This representation of the partition function was studied (for the non-perturbed theory) by Boulatov and Kazakov [12]. Note that the absolute value can be abandoned, since the integrand is homogeneous.

The grand canonical partition function can be written as the absolute value of a Fredholm determinant

$$\mathcal{Z}(\mu, t_n) = |\text{Det}(1 + q^{i\mu} \hat{K})| \tag{5.4}$$

where the Fredholm kernel is defined by the contour integral

$$(\hat{K}f)(z) = - \oint \frac{[dz']}{2\pi i} \frac{f(z')}{q^{1/2}z - q^{-1/2}z'}. \tag{5.5}$$

The integral has poles when $z_i = qz_j$ and should be evaluated by adopting a prescription for surrounding the poles. The prescription used in [12] is to add a small imaginary part to R so that $|q| < 1$. In the case of real $q \in [0, 1)$ the partition function was studied by Gaudin [22].

5.2. Evaluation of the partition function of the non-perturbed theory

Boulatov and Kazakov showed in [12] that the prescription for the contour integration gives the same result as (3.18) for the Fredholm determinant (5.4), up to non-perturbative terms $\mathcal{O}(e^{-\pi\mu})$. Here we recall their calculation. The integration kernel (5.5) acts on the monomials z^n as

$$Kz^n = \begin{cases} q^{n+\frac{1}{2}}z^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \tag{5.6}$$

and the Fredholm determinant (5.4) reads

$$\mathcal{Z}(\mu) = \prod_{r>0} (1 + q^{i\mu+r}).$$

The free energy then is equal to an infinite sum

$$\mathcal{F}(\mu) = \sum_{n \geq 0} \log(1 + q^{i\mu+n+1/2}) = \sum_{m \geq 1} \frac{(-)^m q^{im\mu}}{m} \frac{1}{q^{m/2} - q^{-m/2}}. \tag{5.7}$$

The last sum is the result and can be written as the integral

$$\mathcal{F}(\mu) = \int_{\mathcal{C}} \frac{dy}{y} \frac{q^{i\mu y}}{4 \sinh(yR) \sinh(y)}. \tag{5.8}$$

along a contour \mathcal{C} circling around the poles $y = in, n > 0$ only. If $|q| < 1$, then \mathcal{C} goes from $-\infty$ to 0 and then up the imaginary axis to $i\infty$.

Note that if we close the contour around the poles of $\sinh \pi Ry$, then the result will be the sum over the residues $k = inR (n > 0)$ which can be written as the partition function for the dual radius $\tilde{R} = 1/R$.

5.3. Toda integrable structure

The Fredholm determinant (5.4) with non-homogeneous measure (5.3) can be represented as the Fock expectation value in a theory of chiral fermions defined on the unit circle [13]. The deformations by winding modes are introduced as Bogolyubov transformations of the left and right fermionic vacua and the partition function was identified as a τ -function of the Toda lattice hierarchy.

As a consequence, the free energy satisfies an infinite hierarchy of PDE with the Toda ‘times’ \tilde{t}_n . The first one is the Toda equation dual to the equation (4.29):

$$i \frac{\partial}{\partial \tilde{t}_1} \frac{\partial}{\partial \tilde{t}_{-1}} \tilde{\phi}(\mu) = e^{i\tilde{\phi}(\mu) - i\tilde{\phi}(\mu-i)} - e^{i\tilde{\phi}(\mu+i) - i\tilde{\phi}(\mu)} \tag{5.9}$$

where

$$\tilde{\phi}(\mu) = 2 \sin(\partial_\mu/2) \mathcal{F}(\mu, R). \tag{5.10}$$

Using the scaling relation given in the appendix, one can reduce (5.9) to an ordinary differential equation, which should be solved with initial condition (5.8). In the quasi-classical (genus zero) limit, this differential equation can be integrated to an algebraic equation for the string susceptibility $u_0 = \partial_\mu \tilde{\phi}$

$$\mu e^{\frac{1}{R} \partial_\mu \tilde{\phi}} + \tilde{t}_1 \tilde{t}_{-1} (R - 1) e^{\frac{2-R}{R} \partial_\mu \tilde{\phi}} = 1. \tag{5.11}$$

The one- and two-point correlators were calculated by Alexandrov and Kazakov from the higher equations of the Toda hierarchy [16]. These results were later confirmed in [17] using the Lax formalism of the constrained Toda system.

5.4. Lax operators and string equation

We will consider the potential $\tilde{U}(z)$ in the measure (5.3) as the value on the unit circle $\bar{z}z = 1$ of the potential

$$\tilde{U}(z, \bar{z}) = \sum_{n \geq 0} (\tilde{t}_n z^n + \tilde{t}_{-n} \bar{z}^n) \tag{5.12}$$

defined in the whole complex plane. Let us assume that the spectral variables z and \bar{z} are represented (in some sense) by a pair of Lax operators of the form

$$L = e^{-\frac{1}{2}i\tilde{\phi}} \hat{\omega} \left(1 + \sum_{k \geq 1} u_k \hat{\omega}^{1-k} \right) e^{\frac{1}{2}i\tilde{\phi}} \quad \bar{L} = e^{\frac{1}{2}i\tilde{\phi}} \hat{\omega}^{-1} \left(1 + \sum_{k \geq 1} u_k \hat{\omega}^{1-k} \right) e^{-\frac{1}{2}i\tilde{\phi}} \tag{5.13}$$

where the phase $\tilde{\phi}$ is related to the free energy by

$$\tilde{\phi}(\mu) = -i[\mathcal{F}(\mu + i/2) - \mathcal{F}(\mu - i/2)]. \tag{5.14}$$

In order to find the constraint satisfied by L and \bar{L} , we consider the non-perturbed theory, for which the expressions in the parentheses are equal to 1. The functional equation (3.19) means that the bare phase $\tilde{\phi}_0$ satisfies

$$\tilde{\phi}_0(\mu + i/R) - \tilde{\phi}_0(\mu - i/R) = -i \log \mu. \quad (5.15)$$

In the case of no perturbation, this is equivalent to the algebraic relations

$$L^{1/R} \bar{L}^{1/R} + \bar{L}^{1/R} L^{1/R} = \mu \quad (5.16)$$

and

$$[L^{1/R}, \bar{L}^{1/R}] = i. \quad (5.17)$$

The second identity is invariant with respect to the dressing procedure and therefore is satisfied by the general Lax operators (5.13). This is the string equation for the constrained Toda system.

5.5. Quasiclassical limit and relation with the conformal map problem

In the dispersionless limit $\mu \rightarrow \infty$ the two Lax operators define a smooth closed curve γ in the complex plane, whose equation is written in a parametric form as

$$z = L(\omega) \quad \bar{z} = \bar{L}(\omega) \quad (|\omega| = 1). \quad (5.18)$$

For example, in the case when only \tilde{t}_1 and \tilde{t}_{-1} are nonzero, the explicit form of the curve γ is

$$z = e^{\frac{1}{2}\partial_\mu \tilde{\phi}} \omega (1 + \tilde{t}_{-1} e^{\frac{2-R}{2R}\partial_\mu \tilde{\phi}} \omega^{-1})^R \quad \bar{z} = e^{-\frac{1}{2}\partial_\mu \tilde{\phi}} \omega^{-1} (1 + \tilde{t}_1 e^{\frac{2-R}{2R}\partial_\mu \tilde{\phi}} \omega)^R. \quad (5.19)$$

Assuming that the couplings \tilde{t}_n and \tilde{t}_{-n} are complex conjugate, the map (5.18) can be extended to a conformal map from the exterior of the unit disc $|\omega| < 1$ to a connected domain \mathcal{D} with the topology of a disc containing the point $z = \infty$ and bounded by the curve γ . The couplings \tilde{t}_n ($n \neq 0$) and μ can be thought of as a set of coordinates in the space of closed curves. The relation between the conformal maps and the dispersionless Toda hierarchy was studied recently in a series of papers [23–25].

6. Conclusion

In this paper, we explained how the perturbations of the compactified $c = 1$ string theory by momentum or winding modes are described by integrable deformations of a gauged matrix model on a circle. The momentum and winding modes are associated with the collective excitations of the matter and gauge fields. The deformed system is described by a constrained Toda lattice hierarchy. The Lax formalism for the Toda lattice hierarchy allows us to calculate explicitly the free energy and the correlation functions of the electric or magnetic operators for any genus. In particular, the partition function is a τ -function of the Toda lattice hierarchy.

The integrability takes place only in the grand canonical ensemble, in which the size N is a dynamical variable and the string interaction constant is controlled by the chemical potential μ . The partition function is a Fredholm determinant, and not a usual determinant as in the case of the open matrix chains for the $c < 1$ string theories.

It is not likely that the integrability is conserved for perturbations with both momentum and winding modes. Nevertheless, the calculation of the correlation functions of the momentum modes in the presence of a perturbation mode (or vice versa) seems to be performable albeit very difficult. This calculation might help check the hypothesis (related to the FZZ conjecture [10]) that a strong perturbation by winding modes can lead to a curved background with Euclidean horizon [13].

Another problem, also related to the [10] conjecture, is to find the limit $\mu/t_1 t_{-1} \rightarrow 0$ of the matrix model, which is relevant to the sine-Liouville string theory. This limit seems to be subtle because at small μ the non-perturbative effects can enter into the game. In any case, the $\mu \rightarrow 0$ limit of the correlators of the matrix model does not seem to reproduce the results obtained in the sine-Liouville theory [10, 26].

Finally, it would be very interesting to understand the origin of integrability from the point of view of the worldsheet string theory.

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Appendix. Momentum and winding modes in the $c = 1$ Euclidean string theory

In this section, we will recall briefly the worldsheet description of the $c = 1$ Euclidean string theory with compact target space. The elementary excitations in this theory are closed surfaces, or string worldsheets, embedded in a circle of radius R . An embedded surface is defined by metric tensor $g_{ab}(\sigma)$ ($a, b = 1, 2$) and the position $x(\sigma)$ in the target space as functions of the local coordinates $\sigma = \{\sigma_1, \sigma_2\}$. The free energy of the string theory is given by the functional integral over all connected surfaces

$$\frac{1}{\hbar^2} \mathcal{F}(R, \mu, \hbar) = \int \mathcal{D}g_{ab} \mathcal{D}x e^{-\mathcal{S}(g_{ab}, x)} \quad (\text{A.1})$$

with a weight given by the the Polyakov action

$$\mathcal{S}(g_{ab}, x) = \frac{1}{4\pi} \int_{\text{worldsheet}} d^2\sigma \sqrt{\det g} [g^{ab} \partial_a x \partial_b x + 4\pi \mu + \hat{\mathcal{R}}^{(2)} \log \hbar] \quad (\text{A.2})$$

where $\hat{\mathcal{R}}$ is the local curvature associated with the metric g_{ab} . The parameter μ is the cosmological constant, coupled to the area of the worldsheet, and \hbar is the string coupling constant, which is associated with the processes of splitting and joining of closed strings. By the Euler theorem the global curvature is related to the genus h of the worldsheet as

$$\frac{1}{4\pi} \int d^2\sigma \sqrt{\det g} \hat{\mathcal{R}}^{(2)} = 2 - 2h \quad (\text{A.3})$$

and one can write the free energy as a series

$$\mathcal{F}(R, g) = \sum_{h \geq 0} \hbar^{2h} \mathcal{F}^{(h)}(R, \mu). \quad (\text{A.4})$$

In the conformal gauge $g_{ab} = e^{-2\phi(\sigma)} \delta_{ab}$, the conformal factor ϕ becomes a dynamical field due to the conformal anomaly, and the worldsheet action becomes essentially a $c = 1$ conformal field theory coupled to a Liouville field

$$\mathcal{S} = \frac{1}{4\pi} \int_{\text{worldsheet}} d^2\sigma [(\partial x)^2 + (\partial \phi)^2 + \mu e^{b\phi} + \hat{\mathcal{R}}^{(2)} (\log \hbar + Q\phi) + \text{ghosts}]. \quad (\text{A.5})$$

The background charge Q and the exponent b of the Liouville field are determined by the requirement that the total conformal anomaly vanishes

$$c_{\text{matter}} + c_{\text{Liouville}} + c_{\text{ghosts}} = 1 + (1 + 6Q^2) - 26 = 0 \quad (\text{A.6})$$

and that the perturbation due to the cosmological term $\mu e^{b\phi}$ is marginal

$$\frac{b(2Q - b)}{4} = 1. \quad (\text{A.7})$$

These two conditions give

$$Q = -2 \quad b = -2. \quad (\text{A.8})$$

(The value $Q = 2$ does not lead to a sensible classical limit.) The invariance of the action with respect to shifts $\phi \rightarrow \phi + \phi_0$ implies that the free energy depends on μ and \hbar through the dimensionless combination $\mu\hbar$, which is the statement of the double scaling limit in the $c = 1$ string theory. Therefore we are free to choose $\hbar = 1$ and write the genus expansion (A.4) as an expansion in $1/\mu^2$.

The primary operators associated with the matter field $x(\sigma)$ are the vertex operators $V_e(\sigma) \sim e^{-ie x(\sigma)}$ and the Kosterlitz–Thouless vortices $\tilde{V}_m(\sigma)$. The operator $V_m(\sigma)$ is associated with a discontinuity $2\pi m$ of the field x around the point σ on the worldsheet. We call e and m electric and magnetic charges, in analogy with the Coulomb gas on the plane. The electric and magnetic charges should satisfy the Dirac condition $em = 2\pi \times$ integer. From the point of view of the compactified string theory observables, the electric and magnetic charges are the momentum and winding numbers, correspondingly. The spectrum of the electric and magnetic charges for a periodic target space $x + 2\pi R \equiv x$ is

$$e = n/R \quad m = nR \quad (n \in \mathbb{Z}). \quad (\text{A.9})$$

If we write the position field x as a sum of a holomorphic and an antiholomorphic part, $x = x_R + x_L$, then the Kosterlitz–Thouless vortices are described by the vertex operators for the dual field $\tilde{x} = x_R - x_L$, whose target space is the circle of radius $1/R$. When integrated over the worldsheet, these operators should be accompanied by nontrivial Liouville factors compensating their anomalous dimensions. The integrated operators have the form

$$V_e \sim \int d^2\sigma e^{-ie x} e^{(e-2)\phi} \quad \tilde{V}_m \sim \int d^2\sigma e^{-im\tilde{x}} e^{(m-2)\phi}. \quad (\text{A.10})$$

The Liouville exponents are determined by the condition that the integrands are densities.

We are interested in deformations of the string theory obtained by allowing electric charges $e = n/R$ with fugacities t_n and magnetic charges $m = nR$ with fugacities \tilde{t}_n . This is achieved by adding to the action (A.5) the perturbation term

$$\delta S = \sum_{n \neq 0} (t_n V_{n/R} + \tilde{t}_n \tilde{V}_{nR}). \quad (\text{A.11})$$

The translational invariance of the functional measure $\mathcal{D}\phi$ yields the following Ward identity for the free energy

$$-2\hbar \frac{\partial \mathcal{F}}{\partial \hbar} - 2\mu \frac{\partial \mathcal{F}}{\partial \mu} + \sum_{n \neq 0} (nR - 2) t_n \frac{\partial \mathcal{F}}{\partial t_n} + \sum_{n \neq 0} \left(\frac{n}{R} - 2 \right) \tilde{t}_n \frac{\partial \mathcal{F}}{\partial \tilde{t}_n} = 0. \quad (\text{A.12})$$

This means that the couplings t_n , \tilde{t}_n and the string coupling \hbar scale with respect to the cosmological coupling μ

$$t_n \sim \mu^{1 - \frac{1}{2}|n|/R} \quad \tilde{t}_n \sim \mu^{1 - \frac{1}{2}R|n|} \quad \hbar \sim \mu^{-1}. \quad (\text{A.13})$$

The T-duality symmetry of the original theory

$$x \leftrightarrow \tilde{x} \quad R \rightarrow 1/R \quad \mu \rightarrow \mu/R \quad (\text{A.14})$$

holds for the perturbed theory if one also exchanges the couplings as $t_n \leftrightarrow \tilde{t}_n$ (up to a rescaling by an R -dependent factor).

In the perturbation (A.11) we should retain only the relevant charges $|e| < 2/R$ and $|m| < 2R$. However, the correlation functions of a finite number of irrelevant operators are perfectly meaningful. The interesting phase of the deformed theory, which can be thought of as Coulomb gas coupled to quantum gravity, is the plasma phase, where the fugacities of the charges are tuned so that the Debye length is of the order of the size of the 2D universe.

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